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A CONVEX PARAMETRIZATION OF SUBOPTIMAL H^∞ CONTROLLERS

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A Convex Parametrization of Suboptimal H_∞ Controllers

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Abstract: A new parametrization is proposed for suboptimal H_∞ controllers of order no larger than the plant order. Here such controllers are generated from pairs of symmetric matrices (X, Y) constrained by two Riccati matrix inequalities and some positivity requirements. Interestingly, the Riccati expressions and the positivity conditions are exactly those arising in the usual state-space solution of suboptimal H_∞ problems.

When working with the inverses R and S of X and Y , respectively, the constraints can be rewritten as linear matrix inequalities which define a convex parameter set. This sets up a convenient framework to handle design objectives which can be reflected in terms of (R, S) . Examples of such objectives include reduced-order H_∞ design and the avoidance of pole/zero cancellation between the plant and the controller.

Une Paramétrisation Convexe des Contrôleurs H_∞ Sous-Optimaux

Résumé: Cet article introduit une nouvelle paramétrisation des contrôleurs H_∞ sous-optimaux d'ordre au plus égal à celui du système à commander. Ici les paramètres sont des paires de matrices (X, Y) contraintes par des inégalités de Riccati et des contraintes de positivité. Il y a une analogie totale entre cette caractérisation et celle des performances H_∞ atteignables [3].

En travaillant avec les inverses R et S de X et Y , les contraintes deviennent convexes. Cette propriété est favorable à l'application d'algorithmes d'optimisation convexe avec convergence garantie. Deux applications sont abordées: la synthèse de contrôleurs H_∞ d'ordre réduit et l'élimination des simplifications pôle/zéro entre le système et le contrôleur.

1 Introduction

DGKF's state-space results [3] offer simple and numerically appealing means of designing H_∞ controllers. Indeed, computations essentially reduce to solving the well-known matrix Riccati equation. Moreover, explicit state-space formulas are given for some particular H_∞ controller called the "central controller." Finally, all suitable controllers are parametrized via a linear fractional transformation depending on a free dynamical parameter $Q(s)$ [3]. These attractive features all contribute to the growing popularity of state-space-oriented H_∞ synthesis.

Yet, the diversity in H_∞ controllers is hardly exploited in DGKF's state-space approach. As a result, applications make exclusive use of the central controller in spite of certain undesirable properties. For instance, this particular controller tends to cancel stable plant poles [10]. In mixed-sensitivity problems with flexible modes, this often results in unacceptable designs from an engineering point of view. Also, the order of the central controller matches that of the augmented plant and may therefore be quite high. Finally, the central controller is the optimal choice with respect to some entropy criterion [6], but may well prove a bad choice with respect to more desirable design objectives. Such objectives include the H_2 norm of certain transfer functions and the time-domain behavior (overshoot, rise- and settling time).

The Q -parametrization of all suboptimal H_∞ controllers could help select controllers which are better-suited to the overall design requirements. Unfortunately, the linear-fractional nature of this parametrization does not blend nicely with the state-space formulation. As a result, there is no obvious way of choosing the free parameter $Q(s)$ in order to, e.g., reduce the controller order or prevent plant/controller cancellations.

This paper introduces a state-space counterpart of the Q -parametrization which is appealing both on computational and analytical grounds. Here the parameter set consists of pairs of symmetric matrices constrained by Riccati inequalities and positivity requirements. Interestingly, these constraints completely parallel DGKF's characterization of feasible γ 's, except that Riccati equations are replaced by Riccati inequalities. To be specific, recall that the "standard" suboptimal H_∞ problem of parameter γ considered in [3] is solvable if and only if the two Riccati equations

$$A^T X + X A + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 = 0; \quad (1.1)$$

$$A Y + Y A^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T = 0 \quad (1.2)$$

have stabilizing solutions X_∞ and Y_∞ which further satisfy:

$$X_\infty \geq 0; \quad Y_\infty \geq 0; \quad \rho(X_\infty Y_\infty) < \gamma^2. \quad (1.3)$$

In comparison, this paper shows that γ -suboptimal controllers of order no larger than the plant order can be "parametrized" by the set of pairs (X, Y) of symmetric matrices constrained by:

$$\begin{cases} A^T X + X A + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 < 0 \\ A Y + Y A^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T < 0 \\ X > 0, \quad Y > 0, \quad \rho(XY) \leq \gamma^2 \end{cases} \quad (1.4)$$

There is an obvious analogy between the two results. In fact, DGKF's solvability conditions are easily deduced from our parametrization (see Subsection 4.3 below). Note that the requirement $\rho(XY) < \gamma^2$ is relaxed to $\rho(XY) \leq \gamma^2$, equality corresponding to reduced-order H_∞ controllers.

The paper is organized as follows. The next section gives a precise statement of the suboptimal H_∞ problem and recalls the instrumental Bounded Real Lemma. Section 3 is an intermediate step

toward the advertised parametrization. There, insight is gained into the connection between suboptimal H_∞ controllers and pairs (X, Y) satisfying (1.4). This will prove useful for the reconstruction of H_∞ controllers from such (X, Y) 's. Section 4 presents the main result which is a parametrization of suboptimal H_∞ controllers by the set of pairs (X, Y) satisfying (1.4). The constraints (1.4) are shown to define a convex set when reformulated in terms of $R := X^{-1}$ and $S := Y^{-1}$. Since convexity is highly desirable in optimization problems, all theorems are stated in terms of R, S instead of X, Y . A simple algorithm is also proposed for the construction of suboptimal controllers from admissible pairs (R, S) . Only elementary matrix computations are involved in this construction. In Section 5 finally, two applications illustrate the potential of this parametrization as a design tool. Firstly, the problem of preventing plant/controller cancellations in H_∞ design is turned into a convex optimization problem in the (R, S) -space. Secondly, applications to reduced-order H_∞ design are discussed.

2 The Suboptimal H_∞ Control Problem

As usual in H_∞ control problems, consider a proper plant $G(s)$ which maps exogenous inputs w and control inputs u to controlled outputs z and measured outputs y . That is, $\begin{pmatrix} z \\ y \end{pmatrix} = G(s) \begin{pmatrix} w \\ u \end{pmatrix}$ where

$$G(s) = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (sI - A)^{-1} (B_1, B_2). \quad (2.1)$$

This realization is taken minimal and n denotes its order ($A \in \mathbb{R}^{n \times n}$). The vectors z, y, w , and u are of size p_1, p_2, m_1 , and m_2 , respectively, with the assumption that $m_1 \geq p_2$ and $p_1 \geq m_2$.

The suboptimal H_∞ control problem of parameter γ consists of finding a dynamic real-rational output feedback law $u = K(s)y$ such that:

- the closed-loop system is internally stable,
- the H_∞ norm of the closed-loop transfer function from w to z is strictly less than γ .

Controllers solving this problem (if any) will be called **γ -suboptimal**. Observing that the closed-loop transfer function from w to z is given by the linear fractional transformation (LFT):

$$\mathcal{F}(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}, \quad (2.2)$$

K is γ -suboptimal if and only if the closed-loop system is internally stable and $\|\mathcal{F}(G, K)\|_\infty < \gamma$. The optimal H_∞ attenuation γ_{opt} is defined as the smallest (asymptotically) achievable γ , that is, the infimum of all $\gamma > 0$ for which there exist γ -suboptimal controllers.

The following assumptions on the state-space realization (2.1) of the plant G are made throughout the paper:

- (A1) (A, B_2, C_2) is stabilizable and detectable.
- (A2) $D_{12}^T(D_{12}, C_1) = (I, 0)$ and $D_{21}(D_{21}^T, B_1^T) = (I, 0)$.
- (A3) $D_{22} = 0$ and $D_{11} = 0$.

Along with (A1), full column rank for D_{12} and full row rank for D_{21} are the only hard requirements for the validity of our results. With these assumptions standing, (A2)-(A3) are only introduced for the resulting notational simplification. Note that the customary requirement that G_{12} and G_{21} have no $j\omega$ -axis transmission zero is not needed in our approach.

The parametrization introduced in this paper is state-space oriented and specialized to controllers whose order is no larger than the plant order. Given the state-space realization

$$K(s) = D_K + C_K(sI - A_K)^{-1}B_K; \quad A_K \in \mathbb{R}^{k \times k}; \quad k \leq n \quad (2.3)$$

of some controller K , a (non necessarily minimal) realization of the closed-loop transfer function from w to z is:

$$\mathcal{F}(G, K)(s) = D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl} \quad (2.4)$$

where

$$\begin{aligned} A_{cl} &= \begin{pmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{pmatrix}; & B_{cl} &= \begin{pmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{pmatrix}; \\ C_{cl} &= (C_1 + D_{12} D_K C_2 \quad D_{12} C_K); & D_{cl} &= D_{12} D_K D_{21}. \end{aligned} \quad (2.5)$$

In terms of these state-space parameters, the controller K is γ -suboptimal if and only if

$$A_{cl} \text{ is stable and } \|D_{cl} + C_{cl}(sI - A_{cl})^{-1}B_{cl}\|_\infty < \gamma. \quad (2.6)$$

Interestingly, these two requirements can be lumped into a single condition involving a Riccati inequality of order $n + k$. This instrumental reformulation is known as the Bounded Real Lemma. When expressed in terms of Riccati inequalities, this lemma does not require minimality of the realization and takes the fully general form given next.

Lemma 2.1 (Bounded Real Lemma) *Consider some (non necessarily minimal) realization $T(s) = D + C(sI - A)^{-1}B$ of some (non necessarily square) transfer function $T(s)$. Then the following statements are equivalent:*

- (i) A is stable and $\|D + C(sI - A)^{-1}B\|_\infty < \gamma$,
- (ii) $\sigma_{\max}(D) < \gamma$ and there exists a symmetric positive definite solution X to the ARI:

$$A^T X + X A + \gamma^{-2} C^T C + (\gamma^{-2} C^T D + X B)(I - \gamma^{-2} D^T D)^{-1}(\gamma^{-2} C^T D + X B)^T < 0. \quad (2.7)$$

3 Suboptimal Controllers and Riccati Inequalities

In this section, the conditions (2.6) for γ -suboptimality are converted into Riccati inequality constraints via the Bounded Real Lemma. The Riccati inequalities involve the plant state-space parameters, but also some of the controller parameters. Hence this reformulation is not readily useful for controller synthesis. Nonetheless, it constitutes an insightful intermediate step toward the convex parametrization announced in Section 1. In particular, it brings out the connection between suboptimal controllers and the constraints (1.4).

The next theorem gives necessary conditions for γ -suboptimality in terms of matrix inequalities involving two matrix parameters R and S . Sufficiency of these conditions is addressed in the subsequent Theorem 3.2. These two results heavily rely on the Bounded Real Lemma and strengthen Theorem 6.5, p. 222 of [9]. Note that X, Y in (1.4) correspond to R^{-1} and S^{-1} , respectively, for reasons which will become clear in Section 4.

Theorem 3.1 (Necessity Part) Assume (A1)-(A3) and suppose the controller $K(s)$ is γ -suboptimal of order no larger than the plant order ($k \leq n$). Let

$$K(s) = D_K + C_K(sI - A_K)^{-1}B_K; \quad A_K \in \mathbb{R}^{k \times k} \quad (3.1)$$

denote a minimal realization of K . Then

(C1) $\sigma_{\max}(D_K) < \gamma$

and there exist symmetric matrices R, S in $\mathbb{R}^{n \times n}$ and M, N in $\mathbb{R}^{n \times k}$ such that

(C2) R, S, M , and N satisfy

$$\Delta_R := AR + RA^T + RC_1^T C_1 R + \gamma^{-2} B_1 B_1^T - B_2 B_2^T + M_R(I - \gamma^{-2} D_K D_K^T)^{-1} M_R^T < 0; \quad (3.2)$$

$$\Delta_S := A^T S + SA + SB_1 B_1^T S + \gamma^{-2} C_1^T C_1 - C_2^T C_2 + N_S(I - \gamma^{-2} D_K^T D_K)^{-1} N_S^T < 0; \quad (3.3)$$

$$MN^T = \gamma^{-2} I - RS \quad (3.4)$$

with the shorthands:

$$M_R := B_2 + RC_2^T D_K^T + MC_K^T; \quad N_S := C_2^T + SB_2 D_K + NB_K. \quad (3.5)$$

(C3) $R > 0$, $S > 0$, and $\lambda_{\min}(RS) \geq \gamma^{-2}$.

Proof: See Appendix B. ■

The matrices R, S, M, N are constructed by applying the Bounded Real Lemma to the realization (2.4) of the closed-loop system $\mathcal{F}(G, K)$. Specifically, given any solution $X_{cl} \in \mathbb{R}^{(n+k) \times (n+k)}$ of the Riccati inequality (2.7) written for $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$, suitable R, S, M, N are readily obtained by partitioning:

$$X_{cl} = \begin{pmatrix} S & N \\ N^T & \star \end{pmatrix}; \quad \gamma^{-2} X_{cl}^{-1} = \begin{pmatrix} R & M \\ M^T & \star \end{pmatrix}; \quad R, S \in \mathbb{R}^{n \times n}, \quad M, N \in \mathbb{R}^{n \times k}. \quad (3.6)$$

Hence (C2) expresses the fact that $\|\mathcal{F}(G, K)\|_\infty < \gamma$ since it has to do with existence of some real symmetric solution X_{cl} to (2.7). Meanwhile, (C3) expresses internal stability of the closed-loop system since it is equivalent to $X_{cl} > 0$. Note that the correspondence $K(s) \rightarrow (R, S)$ is not univocal; because of the inequalities, to each suboptimal $K(s)$ corresponds a family of pairs (R, S) (see Subsection 4.2 for detail).

Concerning the controller order k , note that if $k < n$ then $\lambda_{\min}(RS) = \gamma^{-2}$ or more precisely, $\text{rank}(\gamma^{-2}I - RS) = k$ (recall that $M, N \in \mathbb{R}^{n \times k}$). Equivalently in terms of $(X, Y) := (R^{-1}, S^{-1})$, we have $\rho(XY) = \gamma^2$ and $\text{rank}(\gamma^{-2}XY - I) = k$. As confirmed in the sequel, rank deficiency of $\gamma^{-2}I - RS$ is characteristic of reduced-order H_∞ controllers and the rank of this “coupling matrix” determines the controller order (see Subsection 4.4).

We now turn to the converse of Theorem 3.1. Observe that the controller state matrix A_K does not appear in (C1)-(C3). Hence the main concern is whether the knowledge of R, S, M, N and B_K, C_K, D_K satisfying (C1)-(C3) is sufficient to reconstruct the A_K matrix of some γ -suboptimal controller. The next theorem confirms the feasibility of this operation.

Theorem 3.2 (Sufficiency Part) With the notation and assumptions of Theorem 3.2, consider any matrices $R, S, M, N, B_K, C_K, D_K$ such that

- $R = R^T \in \mathbb{R}^{n \times n}$, $S = S^T \in \mathbb{R}^{n \times n}$, and $M, N \in \mathbb{R}^{n \times k}$ have full column rank ($k \leq n$).
- $R, S, M, N, B_K, C_K, D_K$ jointly satisfy (C1)-(C3).

Then there exists $A_K \in \mathbb{R}^{k \times k}$ for which the controller $K(s) := D_K + C_K(sI - A_K)^{-1}B_K$ is γ -suboptimal and of order at most k . Such a matrix A_K is constructed as follows.

(i) Compute matrices $\Delta_{12} \in \mathbb{R}^{n \times k}$ and $\Delta_{22} \in \mathbb{R}^{k \times k}$ such that $\begin{pmatrix} -\Delta_S & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix} > 0$ and

$$\Delta_R + \gamma^2(R, M) \begin{pmatrix} -\Delta_S & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix} \begin{pmatrix} R \\ M^T \end{pmatrix} = 0. \quad (3.7)$$

(ii) Compute $A_K \in \mathbb{R}^{k \times k}$ as the unique solution of

$$\begin{aligned} -NA_K M^T &= \gamma^{-2}A^T + S(\gamma^{-2}B_1 B_1^T + B_2 C_K M^T) + (\gamma^{-2}C_1^T C_1 + N B_K C_2)R + \\ &S(A + B_2 D_K C_2)R + \gamma^{-2}N_S(I - \gamma^{-2}D_K^T D_K)^{-1}D_K^T M_R^T - \Delta_S R + \Delta_{12} M^T. \end{aligned} \quad (3.8)$$

Proof: See Appendix B. ■

Hence some suitable controller state matrix A_K can be computed by solving two systems of linear equations. Note that A_K is directly sought of dimensions $k \times k$ in the reduced-order case. Consequently, no switching to descriptor form is needed when $\gamma^{-2}I - RS$ or equivalently $I - \gamma^{-2}XY$ is rank-deficient. Interestingly, this scheme also applies to optimal central controller design when $I - \gamma^{-2}X_\infty Y_\infty$ is singular at γ_{opt} [4]. Finally, observe that (3.8) can also be written (with the notation $Z := \gamma^{-2}I - RS$):

$$\begin{aligned} -NA_K M^T &= Z^T(A + RC_1^T C_1)^T + N B_K C_2 R + S\Delta_R - \Delta_S R + \Delta_{12} M^T \\ &\quad \{\gamma^{-2}N B_K D_K^T + Z^T C_2^T D_K^T - SMC_K^T\}(I - \gamma^{-2}D_K D_K^T)^{-1}M_R^T, \end{aligned} \quad (3.9)$$

or equivalently as:

$$\begin{aligned} -NA_K M^T &= (A + B_1 B_1^T S)^T Z^T + S B_2 C_K M^T + \Delta_{12} M^T + \\ &N_S(I - \gamma^{-2}D_K^T D_K)^{-1} \{\gamma^{-2}M C_K^T D_K + Z B_2 D_K - R N B_K\}^T. \end{aligned} \quad (3.10)$$

Simple algebra involving (3.2)-(3.3) accounts for these reformulations (see Appendix B for detail).

At first glance, the trading of the unknown A_K against the four unknowns R, S, M, N seems rather unfavorable. However, much structure has been gained in the process. In the full-order case for instance, M and N are square invertible and can be absorbed in C_K, B_K without loss of generality. The two Riccati inequalities then decouple and given any R, S satisfying (C3) and

$$\begin{aligned} AR + RA^T + RC_1^T C_1 R + \gamma^{-2}B_1 B_1^T - B_2 B_2^T &< 0; \\ A^T S + SA + S B_1 B_1^T S + \gamma^{-2}C_1^T C_1 - C_2^T C_2 &< 0, \end{aligned}$$

some B_K, C_K, D_K can be constructed such that (3.2)-(3.3) hold. This simple observation is the gateway to the state-space parametrization introduced in the next section.

4 Convex Parametrization of Suboptimal Controllers

The results of Section 3 are awkward for design purposes because the unknown controllers parameters B_K, C_K, D_K still appear in (3.2)-(3.3). In this section, M, N, B_K, C_K, D_K are removed altogether from the formulation. This leads to our main result: a parametrization of γ -suboptimal controllers (of order no larger than the plant order) by all pairs (R, S) in the set

$$\mathcal{A}_\gamma := \{(R, S) : R = R^T \text{ and } S = S^T \text{ and } R, S \text{ satisfy the constraints } (C_\gamma)\} \quad (4.1)$$

where

$$(C_\gamma) \quad \begin{cases} AR + RA^T + RC_1^T C_1 R + \gamma^{-2} B_1 B_1^T - B_2 B_2^T < 0 \\ A^T S + SA + SB_1 B_1^T S + \gamma^{-2} C_1^T C_1 - C_2^T C_2 < 0 \\ R > 0, \quad S > 0, \quad \lambda_{\min}(RS) \geq \gamma^{-2} \end{cases} \quad (4.2)$$

Note that (4.2) is equivalent to (1.4) in the transformation $(R, S) \rightarrow (X, Y) := (R^{-1}, S^{-1})$.

Clearly, all pairs (R, S) associated with γ -suboptimal controllers via Theorem 3.1 lie in \mathcal{A}_γ . Conversely, the next theorem shows that given any (R, S) in \mathcal{A}_γ , we can reconstruct some γ -suboptimal controller of order $k \leq n$ by some elementary matrix computations. This confirms the relevance of Riccati inequalities in H_∞ design (see also, e.g., [11, 12]).

Theorem 4.1 *Assume (A1)-(A3) and introduce the Riccati residuals:*

$$\mathcal{R}_R := AR + RA^T + RC_1^T C_1 R + \gamma^{-2} B_1 B_1^T - B_2 B_2^T; \quad (4.3)$$

$$\mathcal{R}_S := A^T S + SA + SB_1 B_1^T S + \gamma^{-2} C_1^T C_1 - C_2^T C_2. \quad (4.4)$$

Given any pair (R, S) in \mathcal{A}_γ , a γ -suboptimal controller $K(s) = D_K + C_K(sI - A_K)^{-1} B_K$ of order no larger than $k := \text{rank}(\gamma^{-2}I - RS)$ can be reconstructed as follows.

- (a) Compute full column rank matrices $M, N \in \mathbb{R}^{n \times k}$ such that $MN^T = \gamma^{-2}I - RS$.
- (b) Select D_K as follows. If $k = n$, pick any matrix satisfying $\sigma_{\max}(D_K) < \gamma$. Otherwise, chose D_K so that $\sigma_{\max}(D_K) < \gamma$ and

$$\begin{pmatrix} I & \gamma^{-1} D_K \\ \gamma^{-1} D_K^T & I \end{pmatrix} > \begin{pmatrix} -B_2^T \\ \gamma C_2 R \end{pmatrix} V_2 \{V_2^T (\gamma^2 RC_2^T C_2 R - \mathcal{R}_R) V_2\}^{-1} V_2^T (-B_2, \gamma RC_2^T) \quad (4.5)$$

where V_2 denotes an orthonormal basis of $\text{Ker}(\gamma^{-2}I - SR)$. A possible choice is

$$D_K = -\gamma^2 B_2^T V_2 \{V_2^T (\gamma^2 RC_2^T C_2 R - \mathcal{R}_R) V_2\}^{-1} V_2^T RC_2^T. \quad (4.6)$$

- (c) For B_K, C_K , chose any matrices compatible with

$$\Delta_R := \mathcal{R}_R + (B_2 + RC_2^T D_K^T + MC_K^T)(I - \gamma^{-2} D_K D_K^T)^{-1} (B_2 + RC_2^T D_K^T + MC_K^T)^T < 0; \quad (4.7)$$

$$\Delta_S := \mathcal{R}_S + (C_2^T + SB_2 D_K + NB_K)(I - \gamma^{-2} D_K^T D_K)^{-1} (C_2^T + SB_2 D_K + NB_K)^T < 0. \quad (4.8)$$

- (d) Use Theorem 3.2 to reconstruct A_K .

Proof: See Appendix C. ■

Summing up, there is an exhaustive correspondence between the set of γ -suboptimal controllers of order $k \leq n$ and the set \mathcal{A}_γ defined by (4.1)-(4.2). Properties and interpretation of this result are discussed in the remainder of the section.

4.1 Properties of \mathcal{A}_γ

The parameter set \mathcal{A}_γ has nice properties which make this formulation computationally appealing. First of all, \mathcal{A}_γ is a convex subset of $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$. This is easily seen when observing that the constraints (C_γ) are equivalent to the three linear matrix inequalities:

$$\begin{pmatrix} AR + RA^T + \gamma^{-2}B_1B_1^T - B_2B_2^T & RC_1^T \\ C_1R & -I \end{pmatrix} < 0; \quad (4.9)$$

$$\begin{pmatrix} A^TS + SA + \gamma^{-2}C_1^TC_1 - C_2^TC_2 & SB_1 \\ B_1^TS & -I \end{pmatrix} < 0; \quad (4.10)$$

$$\begin{pmatrix} R & \gamma^{-1}I \\ \gamma^{-1}I & S \end{pmatrix} \geq 0, \quad (4.11)$$

each of which define a convex subset of $\mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$. Incidentally, convexity does not hold for the set of pairs (X, Y) defined by (1.4). This is our main motivation for using the parameters R, S instead of X, Y .

With γ_{opt} denoting the smallest feasible γ , the family of sets $\{\mathcal{A}_\gamma\}_{\gamma > 0}$ has the following properties:

- $\mathcal{A}_\gamma \neq \emptyset$ if and only if $\gamma > \gamma_{opt}$;
- $\mathcal{A}_{\gamma_1} \subset \mathcal{A}_{\gamma_2}$ whenever $\gamma_1 < \gamma_2$.

In other words, the set \mathcal{A}_γ grows larger as γ increases.

Another interesting property concerns extremal points of \mathcal{A}_γ for $\gamma > \gamma_{opt}$. The next lemma shows that the stabilizing solutions X_∞ and Y_∞ of the usual H_∞ Riccati equations (1.1)-(1.2) are minimizers among all X, Y satisfying (1.4).

Lemma 4.2 *Assume $\gamma > \gamma_{opt}$ and let X_∞ and Y_∞ denote the stabilizing solutions of (1.1)-(1.2), respectively. Then for any pair of $n \times n$ symmetric matrices (X, Y) satisfying*

$$A^TX + XA + X(\gamma^{-2}B_1B_1^T - B_2B_2^T)X + C_1^TC_1 < 0; \quad (4.12)$$

$$AY + YA^T + Y(\gamma^{-2}C_1^TC_1 - C_2^TC_2)Y + B_1B_1^T < 0, \quad (4.13)$$

we have

$$X_\infty < X; \quad Y_\infty < Y. \quad (4.14)$$

Proof: The proof is easily adapted from standard monotonicity results on Riccati solutions (see, e.g., [13]). ■

Defining $R_\infty := X_\infty^{-1}$ and $S_\infty := Y_\infty^{-1}$, it follows that (R_∞, S_∞) is maximal among all (R, S) in \mathcal{A}_γ . That is, $R < R_\infty$ and $S < S_\infty$ for all $(R, S) \in \mathcal{A}_\gamma$. Note that (R_∞, S_∞) always lies on the boundary of \mathcal{A}_γ associated with the two Riccati inequalities. Also, R_∞ (respectively, S_∞) is unbounded whenever X_∞ (respectively, Y_∞) is singular. The set \mathcal{A}_γ is then unbounded toward infinity. To gain geometrical insight into the nature of \mathcal{A}_γ , we now discuss a simple example for a plant of order one.

Example 4.3 Consider the plant $G(s)$ of order one and state-space parameters (cf. (2.5)):

$$A = 1; \quad B_1 = C_1 = 0; \quad B_2 = 1; \quad C_2 = 2; \quad B_2 = 1; \quad D_{12} = D_{21} = 1; \quad D_{11} = D_{22} = 0.$$

The lowest achievable γ for this plant is $\gamma_{opt} = 1$. For $\gamma > 1$, the set \mathcal{A}_γ is defined by the constraints:

$$2r - 1 < 0; \quad 2s - 4 < 0; \quad r > 0; \quad s > 0; \quad rs \geq \gamma^{-2}.$$

In Figure 4.4, this domain is plotted in the (r, s) -plane for $\gamma = 2$. Note that the Riccati inequalities define two strips while the coupling constraint $rs \geq \gamma^{-2}$ selects the region above the hyperbola $rs = \gamma^{-2}$. Since $X_\infty = 2$ and $Y_\infty = 1/2$ independently of γ , $(R_\infty, S_\infty) := (X_\infty^{-1}, Y_\infty^{-1}) = (1/2, 2)$ is an extremal point of \mathcal{A}_γ for all $\gamma > 1$. This corner point corresponds to the central controller design while pairs (r, s) sitting on the hyperbola part of the boundary correspond to reduced-order suboptimal controllers.

As γ decreases from $+\infty$, the region \mathcal{A}_γ shrinks further and further. A particularity of this problem is that the Riccati constraints are independent of γ . As a result, only the boundary associated with $rs \geq \gamma^{-2}$ is moving. As $\gamma \rightarrow \gamma_{opt}$, this boundary approaches the corner point (R_∞, S_∞) and \mathcal{A}_γ shrinks to an empty set as illustrated on Figure 4.5. At $\gamma = \gamma_{opt}$, \mathcal{A}_γ is empty since no controller can internally stabilize the plant while enforcing the strict constraint $\|\mathcal{F}(G, K)\|_\infty < \gamma_{opt}$. Nevertheless, the corner point (R_∞, S_∞) generally yields a solution of the “optimal” H_∞ problem where $\|\mathcal{F}(G, K)\|_\infty = \gamma_{opt}$. Due to the triviality of this example, the central controller design is the only feasible design at the optimum. In more realistic problems however, there often remain degrees of freedom in the choice of (R, S) at γ_{opt} .

4.2 Nature of the Parametrization

The word “parametrization” should be taken in a loose sense here since the correspondence $K(s) \leftrightarrow (R, S)$ is not univocal. Indeed, given any (R, S) in \mathcal{A}_γ we can reconstruct a family of γ -suboptimal controllers where state-space parameters vary in open sets. Conversely, to each suboptimal controller corresponds a convex subset of \mathcal{A}_γ . To see this, recall from Section 3 that S is constructed from solutions X_{cl} of the Bounded Real Lemma inequality (2.7) written for the closed-loop system. For fixed (A_K, B_K, C_K, D_K) , the solution set of this inequality is convex since (2.7) can be rewritten as

$$\begin{pmatrix} A^T X + X A + \gamma^{-2} C^T C & \gamma^{-2} C^T D + X B \\ \gamma^{-2} D^T C + B^T X & \gamma^{-2} D^T D - I \end{pmatrix} < 0$$

which is affine in X . The resulting set of S matrices is therefore convex. Similar conclusions hold for the R component upon observing that $\gamma^{-2} X_{cl}^{-1}$ solves the Riccati inequality dual of (2.7) in the transformation $(A, B, C, D) \rightarrow (A^T, C^T, B^T, D^T)$.

This topological rather than pointwise correspondence between suboptimal controllers and pairs (R, S) may seem a handicap at first. Fortunately, certain important properties of the controller or the closed-loop system can be monitored through (R, S) independently of the particular controller reconstructed (see Section 5). Nevertheless, more research is needed to characterize the family of controllers associated with a given $(R, S) \in \mathcal{A}_\gamma$ and to exploit these additional degrees of freedom.

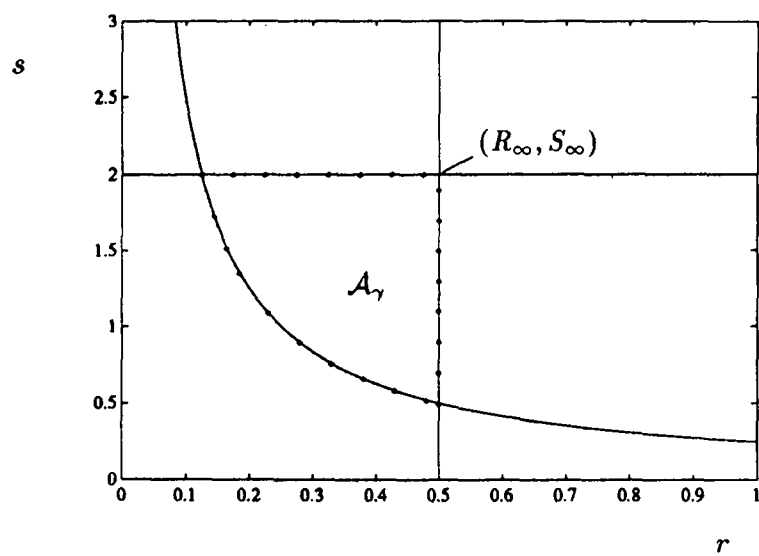


Figure 4.4

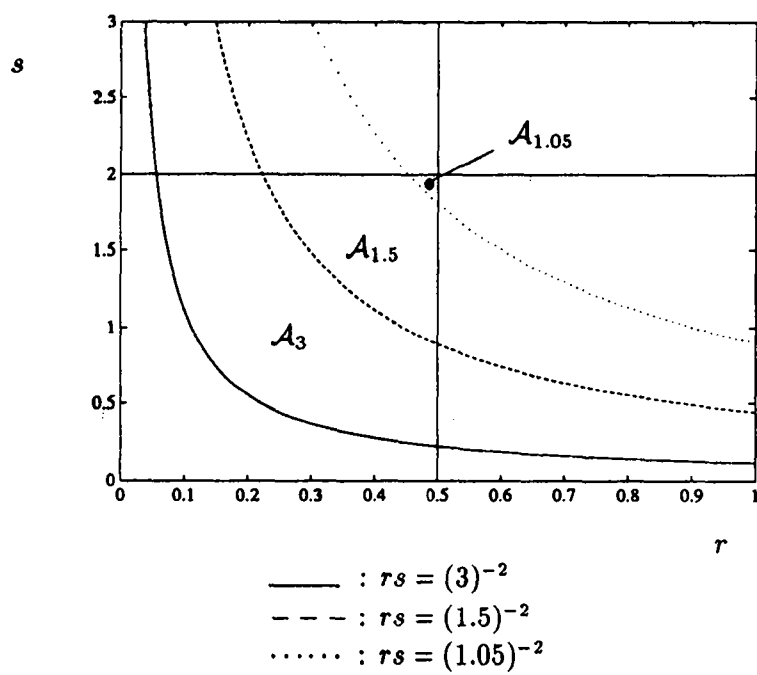


Figure 4.5

4.3 Connection with DGKF's Results

DGKF's characterization of suboptimal γ 's (see Section 1) is easily deduced from the previous results. Indeed, it can be shown that the Riccati equations (1.1)-(1.2) do have stabilizing solution X_∞ and Y_∞ whenever \mathcal{A}_γ is non empty (see [5] for details). Invoking monotonicity results comparable to those of Lemma 4.2, we further have $X_\infty^{-1} \geq R$ and $Y_\infty^{-1} \geq S$ for all (R, S) in \mathcal{A}_γ . Hence X_∞ and Y_∞ satisfy (1.3). As for the converse, it readily follows from continuity properties of stabilizing Riccati solutions under small perturbations (see e.g., [2]). Specifically, if (1.1)-(1.2) have stabilizing solutions then for ϵ small enough, the perturbed equations

$$\begin{aligned} A^T X + X A + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 + \epsilon I &= 0; \\ A Y + Y A^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T + \epsilon I &= 0 \end{aligned}$$

retain stabilizing solutions which continuously depend on ϵ . Consequently, \mathcal{A}_γ is nonempty and there exist suboptimal H_∞ controllers for this γ . Finally, note that the extremal pair (X_∞, Y_∞) corresponds to the central controller. This is best seen on the characterization of Theorem 3.1 once (3.2)-(3.3) are rewritten in terms of X, Y as:

$$\begin{aligned} A^T X + X A + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X + C_1^T C_1 + M_X(I - \gamma^{-2} D_K D_K^T)^{-1} M_X^T &< 0; \\ A Y + Y A^T + Y(\gamma^{-2} C_1^T C_1 - C_2^T C_2) Y + B_1 B_1^T + N_Y(I - \gamma^{-2} D_K^T D_K)^{-1} N_Y^T &< 0 \end{aligned}$$

where

$$M_X := X B_2 + C_2^T D_K^T + \bar{M} C_K^T; \quad N_Y := Y C_2^T + B_2 D_K + \bar{N} B_K; \quad \bar{M} \bar{N}^T = \gamma^{-2} X Y - I.$$

When (X, Y) approaches (X_∞, Y_∞) , the quadratic tail terms must approach zero. At the limit, we have $M_X = 0$ and $N_Y = 0$. That is,

$$X_\infty B_2 + C_2^T D_K^T + \bar{M} C_K^T = 0; \quad Y_\infty C_2^T + B_2 D_K + \bar{N} B_K; \quad \bar{M} \bar{N}^T = \gamma^{-2} X_\infty Y_\infty - I. \quad (4.15)$$

Taking $D_K = 0$ for strict properness and, e.g., $(\bar{M}, \bar{N}) = (I, \gamma^{-2} X_\infty Y_\infty - I)$, the matrices B_K, C_K are easily obtained from (4.15) and A_K can be computed from (3.9) with $\Delta_{12} = 0$. This yields

$$\begin{aligned} A_K &= A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty - (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty C_2^T C_2; \\ B_K &= (I - \gamma^{-2} Y_\infty X_\infty)^{-1} Y_\infty C_2^T; \quad C_K = -B_2^T X_\infty \end{aligned}$$

which are exactly the central controller formulas [3]. More detail can be found in [5]. Finally, the notion of central controller can be extended by allowing D_K to be nonzero. This extension proves useful for the design of nearly optimal central controllers (see [4] for details).

5 Applications

5.1 Preventing Pole/Zero Cancellations between the Plant and the Controller

The central controller has the undesirable property of cancelling all stable poles of the plant which are (A, B_1) -uncontrollable or (C_1, A) -unobservable. This behavior is frequently encountered in mixed-sensitivity design and leads to unacceptable designs in the presence of flexible modes. Cancellations in the mixed-sensitivity context have been thoroughly studied in [10]. Various remedies

have been proposed which generally consist of modifying the criterion to penalize cancellations [8, 10]. Yet no general and direct remedy is available in the usual framework. By contrast, the parametrization introduced above offers direct and numerically tractable means of preventing cancellations of lightly damped modes. Indeed, if all controllers obtained from a given $(R, S) \in \mathcal{A}_\gamma$ involve cancellations, then (R, S) must satisfy one the following:

$$\lambda_{\max}(R) \gg 1; \quad \lambda_{\max}(S) \gg 1; \quad |\lambda_{\max}(\mathcal{R}_R)| \ll 1; \quad |\lambda_{\max}(\mathcal{R}_S)| \ll 1. \quad (5.16)$$

Here \mathcal{R}_R and \mathcal{R}_S denote the Riccati residuals defined in (4.4). “Bad” pairs (R, S) therefore lie near the boundary of \mathcal{A}_γ corresponding to the Riccati inequalities.

A qualitative justification of this claim goes as follows. Consider some $(R, S) \in \mathcal{A}_\gamma$, suppose the norms of R, S are comparable to the scale of the problem, and consider some controller $K(s)$ associated with (R, S) via Theorem 4.1. Assuming for simplicity that $D_{cl} = 0$, $X_{cl} > 0$ determined by (3.6) satisfies the closed-loop Bounded Real Lemma equation

$$A_{cl}^T X_{cl} + X_{cl} A_{cl} + X_{cl} B_{cl} B_{cl}^T X_{cl} + \gamma^{-2} C_{cl}^T C_{cl} + \Delta = 0 \quad (5.17)$$

where $\Delta := \begin{pmatrix} -\Delta_S & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix} > 0$ satisfies (3.7). Recall that cancelled modes are nonminimal modes of (A_{cl}, B_{cl}, C_{cl}) . If $K(s)$ cancels some lightly-damped mode of the plant, A_{cl} is nearly unstable and the solution H of

$$A_{cl}^T H + H A_{cl} + I = 0$$

has large norm [7]. In fact, this is also true of the upper-left $n \times n$ subblock of H because of the particular structure of the nonminimal subspace here. Observing that $X_{cl} \geq \lambda_{\min}(\Delta)H$, it follows that either $\|S\| \gg 1$ or $\lambda_{\min}(\Delta) \ll 1$. Since S was assumed of relatively small norm, Δ must therefore be nearly singular.

Summing up, controllers causing cancellations correspond to nearly singular Δ 's in (5.17) (provided that R, S are of small norm). Now, recall from Theorem 4.1 that Δ is not uniquely determined by (R, S) . Thus, for (R, S) to be unacceptable in the sense that all resulting controllers involve cancellations, all Δ 's compatible with (3.7) must be nearly singular. When R and hence M have small norm, this requires that either Δ_R or Δ_S be nearly singular (cf. (3.7) and Lemma A.1 below). In turn, this must extend to \mathcal{R}_R or \mathcal{R}_S because of the remaining degrees of freedom in the choice of Δ_R, Δ_S given $\mathcal{R}_R, \mathcal{R}_S$. Hence (5.16) holds when all designs obtained via Theorem 4.1 involve cancellations of flexible modes.

Steering clear of such pairs (R, S) can be done in a number of ways. For instance, we can seek “good” pairs (R, S) by solving

$$\min_{(R,S) \in \mathcal{A}_\gamma} \text{Trace}(R + S)$$

while placing a steep barrier on the Riccati inequality constraints. This will drive $\lambda_{\max}(\mathcal{R}_R)$ and $\lambda_{\max}(\mathcal{R}_S)$ away from zero and the criterion will ensure that the norms of R and S remain small. Another possible approach consists of finding the analytic center [1] of the intersection

$$\mathcal{A}_\gamma \cap \{ (R, S) : \text{Trace}(R + S) \leq \text{constant} \}.$$

Here again steep barriers should be used for the Riccati inequality constraints. Note that in both cases the resulting problem is convex and can be handled by standard convex optimization algorithms.

5.2 Reduced-Order Design

Reduced-order H_∞ synthesis is a promising application of the parametrization introduced above. Indeed, γ -suboptimal controllers of order $k < n$ have a simple characterization in this framework: they correspond to pairs (R, S) of \mathcal{A}_γ for which $\text{rank}(\gamma^{-2}I - RS) = k$. Such pairs lie on the boundary of \mathcal{A}_γ attached to the constraint $\lambda_{\min}(RS) \geq \gamma^{-2}$ and saturate this constraint in $n - k$ directions. Hence the reduced-order design problem has a clear formulation in terms of the parameters (R, S) : it consists of decreasing the rank of $\gamma^{-2}I - RS$ as much as possible without leaving \mathcal{A}_γ .

For feasible γ 's and with $\lambda_1(RS) \leq \dots \leq \lambda_{n-k}(RS)$ denoting the $n - k$ smallest eigenvalues of RS , the synthesis of controllers of order $k < n$ amounts to minimizing for $(R, S) \in \mathcal{A}_\gamma$ the criterion:

$$\Psi(R, S) = \sum_{i=1}^{n-k} \lambda_i(RS).$$

Indeed, there exist suboptimal controllers of order k if and only if the global minimum of $\Psi(R, S)$ is $(n - k)\gamma^{-2}$. This objective function Ψ is not convex but in fact concave. Hence global convergence is not guaranteed. Nevertheless, the structural properties of the problem should help monitor gradient descent methods so as to obtain significant order reductions upon convergence.

6 Conclusions

A state-space-oriented parametrization of suboptimal H_∞ controllers of order no larger than the plant order has been introduced. Here parameters are positive solutions of Riccati inequalities subject to a coupling constraint. The formalism parallels DGKF's characterization of feasible γ 's except that Riccati equations become inequalities. Some significant design problems are easily formulated in this new framework and the convexity of the parameter set favors the use of convex optimization techniques.

This algebraic parametrization opens new perspectives for improving H_∞ design by making the most out of the suboptimal controller diversity. Applications to plant/controller cancellations and controller order reduction have been discussed. Further research is however needed to explore and exploit the full potential of this design tool.

Appendix A

This appendix lists a few technical results which are useful in proving the theorems of Sections 3 and 4.

Lemma A.1 *Let P and Q be two positive definite $n \times n$ matrices, $R \in \mathbb{R}^{n \times n}$ be nonsingular, and $M \in \mathbb{R}^{n \times k}$ have full column rank ($k \leq n$). If $k < n$, also assume that $V_2^T(Q - RPR^T)V_2 = 0$ where V_2 denotes any basis of the null space of M^T . Then there exist $X \in \mathbb{R}^{k \times k}$ and $Y \in \mathbb{R}^{n \times k}$ such that*

- $(R, M) \begin{pmatrix} P & Y^T \\ Y & X \end{pmatrix} \begin{pmatrix} R^T \\ M^T \end{pmatrix} = Q.$
- $\begin{pmatrix} P & Y^T \\ Y & X \end{pmatrix}$ is positive definite.

Proof: Up to redefining R and M as $Q^{-1/2}R$ and $Q^{-1/2}M$, respectively, assume without loss of generality that $Q = I$. Also assume V_2 is an orthonormal basis of $\text{Ker } M^T$ and complete it into an orthogonal matrix $V = (V_1, V_2)$. Finally, introduce the notation:

$$\begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} M = \begin{pmatrix} M_1 \\ 0 \end{pmatrix}; \quad \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

To ensure the positive definiteness of $\begin{pmatrix} P & Y^T \\ Y & X \end{pmatrix}$, seek X of the form $Y^T P^{-1} Y + Z$ with $Z > 0$. When pre- and postmultiplied by V^T and V , respectively, the first constraint becomes

$$\begin{pmatrix} (R_1 P + M_1 Y) P^{-1} (R_1 P + M_1 Y)^T + M_1^T Z M_1 & (R_1 P + M_1 Y) R_2^T \\ R_2 (R_1 P + M_1 Y)^T & R_2 P R_2^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad (\text{A.1})$$

Solvability for Y, Z requires that $R_2 P R_2^T = I$ which is precisely ensured by the assumption $V_2^T (Q - R P R^T) V_2 = 0$. Observing that M_1 is square invertible since M is full rank, it follows that $Y := -M_1^{-1} R_1 P$ and $Z := (M_1 M_1^T)^{-1}$ solve (A.1). Equivalently, the original requirements are met when choosing $Y = -M_1^{-1} R_1 P$ and $X = M_1^{-1} (I + R_1 P R_1^T) M_1^{-T}$. ■

Lemma A.2 *Let E, Q be two positive definite matrices and $M \in \mathbb{R}^{n \times k}$ have full column rank. Given an arbitrary matrix L of compatible dimensions, the problem of finding a matrix X such that*

$$(L + MX)E(L + MX)^T < Q. \quad (\text{A.2})$$

is solvable if and only if $V_2^T L E L^T V_2 < V_2^T Q V_2$ whenever $M^T V_2 = 0$.

Proof: It suffices to consider the case where the columns of V_2 span the null space of M^T . As in Lemma A.1, assume without loss of generality that $Q = I$, complete V_2 into an orthogonal matrix $V = (V_1, V_2)$, and introduce the partitions:

$$\begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}; \quad \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} M = \begin{pmatrix} M_1 \\ 0 \end{pmatrix}$$

where M_1 is square invertible. Pre- and postmultiplying by V^T and V , respectively, (A.2) becomes:

$$\begin{pmatrix} (L_1 + M_1 X)E(L_1 + M_1 X)^T & (L_1 + M_1 X)E L_2^T \\ L_2 E(L_1 + M_1 X)^T & L_2 E L_2^T \end{pmatrix} < \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad (\text{A.3})$$

Clearly, solvability requires $L_2 E L_2^T < I$ or equivalently, $V_2^T L E L^T V_2 < V_2^T Q V_2$. Conversely, this condition is also sufficient since (A.3) is then satisfied for $X = -M_1^{-1} L_1$. ■

Lemma A.3 *Consider two symmetric matrices $R, S \in \mathbb{R}^{n \times n}$ such that $\text{rank}(\gamma^{-2}I - RS) = k < n$. Let V_2 denote an orthonormal basis of $\text{Ker}(\gamma^{-2}I - SR)$ and introduce the Riccati residuals:*

$$\begin{aligned} \mathcal{R}_R &:= AR + RA^T + RC_1^T C_1 R + \gamma^{-2} B_1 B_1^T - B_2 B_2^T; \\ \mathcal{R}_S &:= A^T S + SA + SB_1 B_1^T S + \gamma^{-2} C_1^T C_1 - C_2^T C_2. \end{aligned}$$

Then

$$V_2^T \{ \mathcal{R}_R + B_2 B_2^T - \gamma^2 R (\mathcal{R}_S + C_2^T C_2) R \} V_2 = 0. \quad (\text{A.4})$$

Proof: From the definition of \mathcal{R}_R and \mathcal{R}_S ,

$$\mathcal{R}_R S - R \mathcal{R}_S = (A + RC_1 C_1^T)(RS - \gamma^{-2}I) - (RS - \gamma^{-2}I)(A + B_1 B_1^T S) + RC_2^T C_2 - B_2 B_2^T S.$$

Pre- and post-multiplying by V_2^T and RV_2 , respectively, and recalling that $SRV_2 = \gamma^{-2}V_2$, it follows that

$$V_2^T(\gamma^{-2}\mathcal{R}_R - R\mathcal{R}_S R)V_2 = V_2^T(RC_2^T C_2 R - \gamma^{-2}B_2 B_2^T)V_2$$

which is exactly (A.4). ■

Appendix B

Proof of Theorems 3.1 and 3.2:

Necessity: Suppose the controller $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ is γ -suboptimal and apply the Bounded Real Lemma 2.1 to the realization (2.5) of the closed-loop transfer function $\mathcal{F}(G, K)$. It follows that $\sigma_{\max}(D_{cl}) < \gamma$ which is equivalent to (C1) in virtue of (A2). In addition, the Riccati equation

$$A_{cl}^T X_{cl} + X_{cl} A_{cl} + \gamma^{-2} C_{cl}^T C_{cl} + (\gamma^{-2} C_{cl}^T D_{cl} + X_{cl} B_{cl})(I - \gamma^{-2} D_{cl}^T D_{cl})^{-1} (\gamma^{-2} C_{cl}^T D_{cl} + X_{cl} B_{cl})^T + \Delta = 0 \quad (\text{B.1})$$

must have a solution $X_{cl} > 0$ for some $\Delta > 0$. This algebraic Riccati inequality (ARI) will readily yield (3.2)-(3.3) upon rewriting it in terms of the plant and controller parameters.

To see this, partition X_{cl} and $\gamma^{-2}X_{cl}^{-1}$ conformably to (2.5) as

$$X_{cl} = \begin{pmatrix} S & N \\ N^T & \star \end{pmatrix}; \quad \gamma^{-2}X_{cl}^{-1} = \begin{pmatrix} R & M \\ M^T & \star \end{pmatrix}; \quad R, S \in \mathbb{R}^{n \times n}; \quad M, N \in \mathbb{R}^{n \times k}. \quad (\text{B.2})$$

With this representation, $X_{cl} > 0$ is equivalent to (C3) and $MN^T = \gamma^{-2}I - RS$. To establish (3.2)-(3.3) and complete the proof, we could proceed as in [5] and block-partition (B.1) as well as its dual solved by $\gamma^{-2}X_{cl}^{-1}$. The (1,1) blocks of these partitioned ARI's would then yield (3.2)-(3.3). With the sufficiency part in mind, it is however preferable to follow a slightly different route. Specifically, observe that R, S, M, N fully determine X_{cl} as the unique solution of $\Pi_2 = X_{cl} \Pi_1$ where

$$\Pi_1 := \begin{pmatrix} I & \gamma R \\ 0 & \gamma M^T \end{pmatrix}; \quad \Pi_2 := \begin{pmatrix} S & \gamma^{-1}I \\ N^T & 0 \end{pmatrix}. \quad (\text{B.3})$$

That is, $X_{cl} = \Pi_2 \Pi_1^\dagger$. When pre- and post-multiplied by Π_1^T and Π_1 , respectively, (B.1) thus becomes:

$$\Pi_1^T A_{cl}^T \Pi_2 + \Pi_2^T A_{cl} \Pi_1 + \gamma^{-2} \Pi_1^T C_{cl}^T C_{cl} \Pi_1 + (\gamma^{-2} \Pi_1^T C_{cl}^T D_{cl} + \Pi_2^T B_{cl})(I - \gamma^{-2} D_{cl}^T D_{cl})^{-1} (\gamma^{-2} \Pi_1^T C_{cl}^T D_{cl} + \Pi_2^T B_{cl})^T + \Pi_1^T \Delta \Pi_1 = 0. \quad (\text{B.4})$$

Note that (B.1) and (B.4) are equivalent since M^T and hence Π_1 are full row rank.

Next, replace $A_{cl}, B_{cl}, C_{cl}, D_{cl}, \Pi_1, \Pi_2$ with their expressions in (2.5) and (B.3), expand the matrix products, and simplify the resulting block expressions. Introducing the notation (3.5) and $\Upsilon_K := I - \gamma^{-2}D_K^T D_K$, simple algebra based on (A2) shows that

$$\gamma^{-2} \Pi_1^T C_{cl}^T D_{cl} + \Pi_2^T B_{cl} = \begin{pmatrix} S B_1 + (N_S - C_2^T \Upsilon_K) D_{21} \\ \gamma^{-1} \{B_1 + M_R D_K D_{21}\} \end{pmatrix}.$$

Together with (A2), this allows to simplify the quadratic term in (B.4) to:

$$(\gamma^{-2}\Pi_1^T C_{cl}^T D_{cl} + \Pi_2^T B_{cl})(I - \gamma^{-2}D_{cl}^T D_{cl})^{-1}(\gamma^{-2}\Pi_1^T C_{cl}^T D_{cl} + \Pi_2^T B_{cl})^T =$$

$$\begin{pmatrix} SB_1 B_1^T S + N_S \Upsilon_K^{-1} N_S^T - N_S C_2 - C_2^T N_S^T + C_2^T \Upsilon_K C_2 & \gamma^{-1} \{SB_1 B_1^T + (N_S \Upsilon_K^{-1} - C_2^T) D_K^T M_R^T\} \\ \gamma^{-1} \{B_1 B_1^T S + M_R D_K (\Upsilon_K^{-1} N_S^T - C_2)\} & \gamma^{-2} \{B_1 B_1^T + M_R D_K \Upsilon_K^{-1} D_K^T M_R^T\} \end{pmatrix}. \quad (\text{B.5})$$

Completing the calculations with Δ partitioned as $\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix}$, it follows that (B.4) and hence (B.1) are equivalent to the following system of three matrix equations:

$$AR + RA^T + RC_1^T C_1 R + \gamma^{-2} B_1 B_1^T - B_2 B_2^T + M_R (I - \gamma^{-2} D_K D_K^T)^{-1} M_R^T + \gamma^2 (R, M) \Delta \begin{pmatrix} R \\ M^T \end{pmatrix} = 0; \quad (\text{B.6})$$

$$A^T S + SA + SB_1 B_1^T S + \gamma^{-2} C_1^T C_1 - C_2^T C_2 + N_S (I - \gamma^{-2} D_K^T D_K)^{-1} N_S^T + \Delta_{11} = 0; \quad (\text{B.7})$$

$$N A_K M^T + S(A + B_2 D_K C_2)R + S(\gamma^{-2} B_1 B_1^T + B_2 C_K M^T) + (\gamma^{-2} C_1^T C_1 + N B_K C_2)R + \gamma^{-2} A^T + \gamma^{-2} N_S (I - \gamma^{-2} D_K^T D_K)^{-1} D_K^T M_R^T + \Delta_{11} R + \Delta_{12} M^T = 0. \quad (\text{B.8})$$

Recalling that $\Delta > 0$, the ARI's (3.2)-(3.3) readily follow from (B.6)-(B.7). Hence (B.1) provides matrices R, S, M, N which satisfy (C2)-(C3).

Sufficiency: Conversely, suppose we are given matrices $R, S, M, N, B_K, C_K, D_K$ which jointly satisfy (C1)-(C3). Without loss of generality, M and N can be assumed full column rank. Our goal is to reconstruct some γ -suboptimal controller from this data. In light of the necessity part, a natural approach consists of

- (i) constructing $\Delta > 0$ for which (B.6)-(B.7) hold;
- (ii) finding a solution $A_K \in \mathbb{R}^{k \times k}$ to (B.8).

When feasible, this will ensure that $X_{cl} := \begin{pmatrix} S & \gamma^{-1} I \\ N^T & 0 \end{pmatrix} \begin{pmatrix} I & \gamma R \\ 0 & \gamma M^T \end{pmatrix}^\dagger$ solves the Bounded Real Lemma equation (B.1). Since this X_{cl} is positive definite in virtue of (C3), Lemma 2.1 will in turn guarantee that $K(s) := D_K + C_K(sI - A_K)^{-1} B_K$ is a k th-order γ -suboptimal controller. Hence it suffices to establish the feasibility of (i)-(ii).

Feasibility of (i): From (B.6) and (B.7), adequate Δ 's should satisfy (with the partition $\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{12}^T & \Delta_{22} \end{pmatrix}$):

$$\Delta > 0; \quad \Delta_R + \gamma^2 (R, M) \Delta \begin{pmatrix} R \\ M^T \end{pmatrix} = 0; \quad \Delta_S + \Delta_{11} = 0. \quad (\text{B.9})$$

Take $\Delta_{11} = -\Delta_S$. Feasibility of the first two constraints in (B.9) is then addressed by Lemma A.1 with $P := -\Delta_S$ and $Q := -\Delta_R$. Since $-\Delta_R > 0$, $-\Delta_S > 0$, and M is full column rank, this lemma is applicable provided that

$$V_2^T (\Delta_R - \gamma^2 R \Delta_S R) V_2 = 0 \quad (\text{B.10})$$

where V_2 denotes any orthonormal basis of $\text{Ker } M^T = \text{Ker } (\gamma^{-2} I - SR)$. To check this condition, observe from Lemma A.3 that

$$V_2^T \{ \mathcal{R}_R + B_2 B_2^T - \gamma^2 R (\mathcal{R}_S + C_2^T C_2) R \} V_2 = 0 \quad (\text{B.11})$$

holds for the Riccati residuals

$$\mathcal{R}_R = \Delta_R - M_R(I - \gamma^{-2}D_K D_K^T)^{-1}M_R^T; \quad \mathcal{R}_S = \Delta_S - N_S(I - \gamma^{-2}D_K^T D_K)^{-1}N_S^T.$$

Using the identities $V_2^T M_R = V_2^T (B_2 + RC_2^T D_K^T)$ and $V_2^T R N_S = V_2^T (RC_2 + \gamma^{-2}B_2 D_K)$, (B.10) is easily deduced by expanding (B.11) and simplifying the resulting expression. Hence Lemma A.1 is applicable and guarantees the existence of some matrix Δ satisfying (B.9).

Feasibility of (ii): Given any Δ solving (B.9), (B.6)-(B.7) hold. To exhibit some controller state matrix $A_K \in \mathbb{R}^{k \times k}$ for which the Bounded Real Lemma equation (B.1) holds, it now suffices to show that the third equation (B.8) is solvable for A_K . Note that (B.8) is exactly (3.8) since $\Delta_{11} = -\Delta_S$.

When $k = n$ (full-order controller), solvability of (B.8) is immediate since M and N are invertible. By contrast, the reduced-order case ($k < n$) deserves some attention due to the row rank deficiency of M and N . Let T denote the right-hand side of (3.8):

$$T := \gamma^{-2}A^T + S(\gamma^{-2}B_1 B_1^T + B_2 C_K M^T) + (\gamma^{-2}C_1^T C_1 + N B_K C_2)R + \\ S(A + B_2 D_K C_2)R + \gamma^{-2}N_S(I - \gamma^{-2}D_K^T D_K)^{-1}D_K^T M_R^T - \Delta_S R + \Delta_{12}M^T$$

and consider some basis V_2 of $\text{Ker}(\gamma^{-2}I - SR) = \text{Ker} M^T$. Observing that RV_2 is then a basis of $\text{Ker}(\gamma^{-2}I - RS) = \text{Ker} N^T$, it is immediate that (B.8) is solvable if and only if

$$TV_2 = 0; \quad V_2^T R T = 0.$$

Checking these requirements is best done on the reformulations (3.9)-(3.10) of (B.8). To obtain (3.10), simply replace the term $SAR + \gamma^{-2}SB_1 B_1^T$ of (B.8) by

$$-S \left\{ RA^T + RC_1^T C_1 R - B_2 B_2^T + M_R(I - \gamma^{-2}D_K D_K^T)^{-1}M_R^T + \gamma^2(R, M)\Delta \begin{pmatrix} R \\ M^T \end{pmatrix} \right\}$$

in virtue of (B.6). A similar manipulation based on (B.7) yields (3.9).

Using the right-hand side of (3.10) as equivalent expression of T , the condition $TV_2 = 0$ is readily verified when observing that $M^T V_2 = 0$, $N^T R V_2 = 0$, and $ZV_2 = 0$. Similarly, (3.9) allows to write

$$V_2^T R T = -V_2^T R S M C_K^T (I - \gamma^{-2}D_K D_K^T)^{-1}M_R^T + V_2^T R (S\Delta_R - \Delta_S R + \Delta_{12}M^T)$$

which also evaluates to zero since $V_2^T R S M = \gamma^{-2}V_2^T M = 0$ and

$$\begin{aligned} S\Delta_R - \Delta_S R + \Delta_{12}M^T &= -\gamma^2 S(R, M)\Delta \begin{pmatrix} R \\ M^T \end{pmatrix} - \Delta_S R + \Delta_{12}M^T \\ &= -\gamma^2 S \{ -R\Delta_S R + R\Delta_{12}M^T + M\Delta_{12}^T R + M\Delta_{22}M^T \} - \Delta_S R + \Delta_{12}M^T \\ &= \gamma^2 (\gamma^{-2}I - SR)(-\Delta_S R + \Delta_{12}M^T) + \gamma^2 SM(\Delta_{12}^T R + \Delta_{22}M^T). \end{aligned}$$

Hence (B.8) is always solvable and uniquely determines A_K since M and N are full column rank. ■

Appendix C

Proof of Theorem 4.1: Assume $(R, S) \in \mathcal{A}_\gamma$. First consider the full-order case ($k = n$) and let D_K be any matrix such that $\sigma_{\max}(D_K) < \gamma$. Since $\mathcal{R}_R, \mathcal{R}_S$ are negative definite and M, N are

square invertible, B_K and C_K can always be chosen so that both Δ_R and Δ_S are negative definite. Invoking Theorem 3.2, γ -suboptimal controllers $K(s) = D_K + C_K(sI - A_K)^{-1}B_K$ can then be derived by reconstructing their state matrix $A_K \in \mathbb{R}^{n \times n}$.

The reduced-order case ($k < n$) is more delicate because M, N are not invertible but only full column rank. As a result, M_R and N_S cannot be arbitrarily set through the choice of B_K and C_K . In turn, this constrains the choice of D_K through (4.7)-(4.8). To be specific, introduce some orthonormal basis V_2 of $\text{Ker}(\gamma^{-2}I - SR)$ as in Lemma A.3. Then the columns of V_2 and RV_2 span $\text{Ker } M^T$ and $\text{Ker } N^T$ and Lemma A.2 shows that (4.7)-(4.8) is feasible for some B_K, C_K if and only if D_K satisfies

$$V_2^T \{ \mathcal{R}_R + (B_2 + RC_2^T D_K^T)(I - \gamma^{-2}D_K D_K^T)^{-1}(B_2 + RC_2^T D_K^T)^T \} V_2 < 0; \quad (\text{C.1})$$

$$V_2^T R \{ \mathcal{R}_S + (C_2^T + SB_2 D_K)(I - \gamma^{-2}D_K^T D_K)^{-1}(C_2^T + SB_2 D_K)^T \} RV_2 < 0. \quad (\text{C.2})$$

Using the identity (A.4) of Lemma A.3, it is easily verified that these two inequalities are equivalent. Hence (C.1) summarizes the constraint on D_K . Observing that

$$\begin{pmatrix} I & \gamma^{-1}D_K \\ \gamma^{-1}D_K^T & I \end{pmatrix}^{-1} = \begin{pmatrix} (I - \gamma^{-2}D_K D_K^T)^{-1} & -\gamma^{-1}(I - \gamma^{-2}D_K D_K^T)^{-1}D_K \\ -\gamma^{-1}D_K^T(I - \gamma^{-2}D_K D_K^T)^{-1} & (I - \gamma^{-2}D_K^T D_K)^{-1} \end{pmatrix},$$

rewrite (C.1) as

$$V_2^T (\mathcal{R}_R - \gamma^2 RC_2^T C_2 R) V_2 + V_2^T (-B_2, \gamma RC_2^T) \begin{pmatrix} I & \gamma^{-1}D_K \\ \gamma^{-1}D_K^T & I \end{pmatrix}^{-1} \begin{pmatrix} -B_2^T \\ \gamma C_2 R \end{pmatrix} V_2 < 0.$$

By a well-known result on 2×2 symmetric block matrices, this last inequality is equivalent to (4.5).

Summing up, (4.7)-(4.8) are feasible if and only if there exists some D_K satisfying (4.5) together with $\sigma_{\max}(D_K) < \gamma$. Now, we claim that the D_K matrix proposed in (4.6) does satisfy these two requirements. To see this, observe that $\mathcal{R}_R < 0$ implies that

$$\gamma^2 C_2 R V_2 \{ V_2^T (\gamma^2 RC_2^T C_2 R - \mathcal{R}_R) V_2 \}^{-1} V_2^T RC_2^T < I. \quad (\text{C.3})$$

Similarly,

$$B_2^T V_2 \{ V_2^T (\gamma^2 RC_2^T C_2 R - \mathcal{R}_R) V_2 \}^{-1} V_2^T B_2 < I \quad (\text{C.4})$$

since $V_2^T (\gamma^2 RC_2^T C_2 R - \mathcal{R}_R) V_2 = V_2^T (B_2 B_2^T - \gamma^2 R \mathcal{R}_S R) V_2$ from Lemma A.3 and $R \mathcal{R}_S R < 0$. With the shorthand $\Phi = V_2^T (\gamma^2 RC_2^T C_2 R - \mathcal{R}_R) V_2$, it follows that

$$\sigma_{\max}(\gamma^2 B_2^T V_2 \Phi^{-1} V_2^T RC_2^T) \leq \gamma \sigma_{\max}(B_2^T V_2 \Phi^{-1/2}) \sigma_{\max}(\gamma \Phi^{-1/2} V_2^T RC_2^T) < \gamma.$$

Consequently, the expression (4.6) satisfies $\sigma_{\max}(D_K) < \gamma$. Moreover, (4.5) holds for this D_K in virtue of (C.3)-(C.4).

Hence matrices D_K, B_K, C_K can always be computed which satisfy (4.6)-(4.7). The construction of $K(s)$ is then completed as in the full-order case. ■

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